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Gleason's Theorem and Completeness of Inner Product Spaces

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We generalize the result of Hamhalter and Pták showing that an inner product space whose dimension is either a nonmeasurable cardinal or an arbitrary cardinal is complete iff its lattice of strongly closed subspaces possesses at least one state or one completely additive state, respectively. Moreover, we show that this lattice of any separable space possesses many σ-finite measures, and we give the Gleason analogue for them.

1. INTRODUCTION AND PRELIMINARIES

There are many characterizations of the completeness of an inner product space via algebraic-topological properties (Gudder, 1974, 1975; Gudder and Holland, 1975) or by algebraic conditions on the lattice $\mathscr{L}(V)$ of all strongly closed subspaces (Amemiya and Araki, 1966; Holland, 1969).

The characterization of Hamhalter and Pták (1987) is interesting: a separable real inner product space is complete iff $\mathscr{L}(V)$ possesses at least one state.

In the present note we generalize their result to nonseparable inner product spaces. Moreover, we show that any incomplete separable inner product space possesses many σ-finite measures, and we give the Gleason analogue for these measures.

Let V be an inner product space over the field C of real or complex numbers with inner product $(\cdot,\cdot)$. For any subset A of V, $A^\perp$ denotes the set of all $x \in V$ such that $(x, y) = 0$ for all $y \in A$. We shall call A a strongly closed subspace of V if $(A^\perp)^\perp = A$. Then $\mathscr{L}(V)$, the set of all strongly closed subspaces of V, is a complete orthocomplemented lattice with the join $\bigvee$,

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meet $\bigwedge$, and orthocomplementation $\perp$ defined by

$$\bigvee_{t\in T} A_t = \Big(\Big(\mathrm{sp}\Big(\bigcup_{t\in T} A_t\Big)\Big)^{\perp}\Big)^{\perp}, \qquad \bigwedge_{t\in T} A_t = \bigcap_{t\in T} A_t, \qquad A \mapsto A^{\perp}$$

and with the minimal and maximal elements $0=\{0\}$ and V, respectively (here sp denotes the linear span).

$\mathscr{L}(V)$ is said to be orthomodular if, for any pair A, $B\in\mathscr{L}(V)$ with $A\subset B$, we have $B=A\vee(B\wedge A^{\perp})$. Amemiya and Araki (1967) proved that V is complete iff $\mathscr{L}(V)$ is orthomodular.

2. MEASURES AND STATES

By $\bigoplus_{t\in T} A_t$ we mean the join of mutually orthogonal elements $A_t\in\mathscr{L}(V)$, $t\in T$. If $0\neq x\in V$, then by P_x we denote the one-dimensional subspace of V spanned over x.

A mapping $m: \mathscr{L}(V)\to[0,\infty]$ with $m(0)=0$ is said to be (i) a finitely additive measure if $m(M\oplus N)=m(M)+m(N)$; (ii) a measure if $m(\bigoplus_{i=1}^{\infty} M_i)=\sum_{i=1}^{\infty} m(M_i)$; and (iii) a completely additive measure if $m(\bigoplus_{t\in T} M_t)=\sum_{t\in T} m(M_t)$ for any index set T. If, in particular, $m(V)=1$, m is called a finitely additive state, state, completely additive state according to (i)–(iii).

The Gleason theorem (Gleason, 1957) says that if V is a complete separable inner product space of dimension $\neq 2$, then any finite measure m on $\mathscr{L}(V)$ is in one-to-one correspondence with positive Hermitian operators T on V of finite trace via

$$m(M)=\mathrm{tr}(TM), \qquad M\in\mathscr{L}(V) \tag{1}$$

(we identify a subspace M with its orthoprojector P^M on it). Eilers and Horst (1975) and Drisch (1979) proved that the assumption of separability of a complete space V is superfluous when V is of dimension of nonmeasurable cardinality (for definition see below). Maeda (1980) and Dvurečenskij (1987) give the characterizations of measures given by (1).

We say, according to Ulam (1930), that the cardinal I is nonmeasurable if there is no probability measure ν on the power set 2^A of a set A whose cardinality is I, such that $\nu(\{a\})=0$ for any $a\in A$. For example, any finite cardinal $\aleph_0$ (the cardinal of all integers) and $\mathbf{c}$ (the cardinal of all reals, under the continuum hypothesis) are nonmeasurable cardinals.

By the dimension of an inner product space we mean the cardinality of any maximal orthonormal set in V.

Hamhalter and Pták (1987) proved that for a separable inner product space V the existence of a state on $\mathscr{L}(V)$ entails the completeness of V.

Hence, for any state m there is a unique density operator $T \in \mathrm{Tr}(\bar{V})$ such that

$$m(M) = \mathrm{tr}(T\bar{M}), \qquad M \in \mathscr{L}(V) \tag{2}$$

where $\bar{M}$ denotes the completion of M in the complete space $\bar{V}$.

It can be shown that a mapping m on $\mathscr{L}(V)$ defined via (2) has the property

$$m\left(\bigoplus_{i=1}^{\infty} M_i\right) \geq \sum_{i=1}^{\infty} m(M_i) \tag{3}$$

Example 1. Let H be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and let V be the linear manifold of H generated by the vectors $\sum_{n=1}^{\infty} e_n/n$, e_2, $e_3, \ldots$. Then $\{e_n\}_{n=2}^{\infty}$ is a maximal orthonormal set in V, and $V = \bigoplus_{n=2}^{\infty} P_{e_n}$. Put $T = (\cdot, e_1)e_1$ and define m via (2). Then

$$1 = m(V) \neq \sum_{n=2}^{\infty} m(P_{e_n}) = 0$$

It is clear that any separable inner product space is countable-dimensional, but the converse is not true. Indeed, the following example is correct [for details see Gudder (1974) for $\mathbf{m} = \aleph_0$ and Dvurečenskij (1986b) for the general case].

Example 2. Let $\mathbf{m}$ be an infinite cardinal such that $\mathbf{m}^{\aleph_0} > \mathbf{m}$ [in particular, let $\mathbf{m} = \aleph_0$, $\aleph_{n.\omega}$ $(n \geq 1)$, $\aleph_{\omega_\omega}$, etc.]. Then in any Hilbert space of dimension $\mathbf{m}^{\aleph_0}$ there is a dense submanifold V of dimension $\mathbf{m}$ containing no orthonormal basis of H.

Below we extend the result of Hamhalter and Pták (1987) to nonseparable inner product spaces, using some of their ideas.

Lemma 1. Let m be a finitely additive state on $\mathscr{L}(V)$. If $A \subset B$, $A, B \in \mathscr{L}(V)$, then

$$m(B) = m(A \vee (B \wedge A^{\perp})) \tag{4}$$

$$m(A) \leq m(B) \tag{5}$$

Moreover, if for any distinct $A, B \in \mathscr{L}(V)$, $A \subset B$, there is a finitely additive state m such that $m(A) < m(B)$, then $\mathscr{L}(V)$ is orthomodular.

Proof. It is evident that if $A \subset B$, then $A \vee (B \wedge A^{\perp}) \subset B$. A simple calculation shows

$$m(A \vee (B \wedge A^{\perp})) = m(A) + m(B \wedge A^{\perp}) = m(A) + 1 - m(B^{\perp} \vee A)$$

$$= m(A) + 1 - m(B^{\perp}) - m(A) = m(B)$$

which gives (4) and (5) and completes the proof. ■

The basic lemma is the following result of Hamhalter and Pták (1987), which is here presented in a more general form; its proof is identical to that of the original one and therefore is omitted.

Lemma 2. Let V be an arbitrary inner product space. Let v be a unit vector in the completion $\bar{V}$ of V. Then for every $\varepsilon > 0$, there exists a $\delta < 0$ such that the following statement holds: If $w \in V$ is a unit vector such that $\|v - w\| < \delta$, then for any finitely additive state m on $\mathscr{L}(V)$ and for each $A \in \mathscr{L}(V)$ satisfying the properties $v \perp A$, $3 \leq \dim A < \infty$, we have the inequality

$$|m(A \vee P_w) - m(A) - m(P_w)| \leq \varepsilon \tag{6}$$

Theorem 3. Let V be an inner product space whose dimension is an infinite nonmeasurable cardinal. If $\mathscr{L}(V)$ possesses at least one state, then V is complete.

Proof. 1. Let m be a state on $\mathscr{L}(V)$. We show that m is completely additive. Let $M = \bigoplus_{t \in T} M_t$, $M_t \in \mathscr{L}(V)$, $t \in T$. Define a function $\mu: 2^T \to [0, 1]$ via

$$\mu(S) = \begin{cases} m\left(\bigoplus_{t \in S} M_t\right) & \text{if} \quad S \neq \emptyset \\ 0 & \text{if} \quad S = \emptyset \end{cases}$$

Then μ is a probability measure on 2^T. Hence, due to Ulam (1930), there is a countable subset $T_0 \subset T$ such that $\mu(T - T_0) = 0$. Therefore, $m(M_t) = 0$ for any $t \notin T_0$ and

$$m(M) = \mu(T) = \mu(T_0) + \mu(T - T_0) = \sum_{t \in T_0} m(M_t) + \sum_{t \notin T_0} m(M_t)$$

2. Let now $\{e_t: t \in T\}$ be a maximal orthonormal system in a given $B \in \mathscr{L}(V)$ and define $B_0 = \bigoplus_{t \in T} P_{e_t}$. Due to the maximality of $\{e_t: t \in T\}$ and (4), we conclude

$$m(B_0) = m(B) \tag{7}$$

In particular, if $B = V$, then there is unit vector $e \in V$ such that

$$m(P_e) > 0 \tag{8}$$

3. Now we claim to show that $B_0 = B$. Suppose the converse. Then $\bar{B}_0 \neq \bar{B}$ and choose a unit vector $v \in \bar{B}$ which is orthogonal to $\bar{B}_0$. Let e be an element of V with (8). Applying Lemma 2 to $\varepsilon = m(P_e)/2 > 0$ and to $\bar{B}$, we can find a $\delta > 0$ such that, for any unit vector $z \in B$ with $\|z - v\| < \delta$ and any $A \perp v$, $3 \leq \dim A < \infty$, (6) holds. Define a unitary operator $U: V \to V$

such that $Ue = z$ and $Uf = f$ for any $f \perp \mathrm{sp}(\{e, z\}) \in \mathscr{L}(V)$. Then m_1 defined by $m_1(M) = m(U^{-1}(M))$, $M \in \mathscr{L}(V)$, is a state on $\mathscr{L}(V)$.

Let $T_0 = \{t_1, t_2, \ldots\}$ be a countable subset of T such that $m_1(\bigoplus_{t \notin T_0} P_{e_t}) = 0$. Define finite-dimensional subspaces $B_n = \bigoplus_{i=1}^{n} \mathrm{sp}(e_{t_n})$. Hence, there is B_{n_0} such that $m_1(B_{n_0}) > m_1(B_0) - \varepsilon$. Then we have

$$m_1(B) \geq m_1(B_{n_0} \vee P_z) \geq m_1(B_{n_0}) + m_1(P_z) - \varepsilon > m_1(B_0) + \varepsilon - \varepsilon = m_1(B_0)$$

which contradicts (7).

4. Finally, let A and B be two elements of $\mathscr{L}(V)$ with $A \subsetneq B$. Using methods from the second and third parts of the present proof, we may find a state m_2 on $\mathscr{L}(V)$ such that $m_2(A) < m_2(B)$. Due to Lemma 1, $\mathscr{L}(V)$ is orthomodular, and the result of Amemiya and Araki (1966) yields V is complete. The proof is finished. ■

Corollary 4. Example 2 gives a stateless orthocomplemented lattice for a nonseparable inner product space.

Theorem 5. Let V be an infinite-dimensional inner product space. If $\mathscr{L}(V)$ possesses at least one completely additive state, then V is complete.

Proof. The proof follows the ideas of the proof of Theorem 3. ■

Theorem 6. An inner product space V is complete iff $\mathscr{L}(V)$ possesses at least one state having a carrier, i.e., there is an element $M_0 \in \mathscr{L}(V)$ such that $m(N) = 0$ iff $N \perp M_0$.

Proof. We claim to show that m is completely additive. Let $M = \bigoplus_{t \in T} M_t$. Denote, for any integer n, $T_n = \{t \in T\colon m(M_t) \geq 1/n\}$. The finiteness of a state m yields that any T_n is a finite subset of T. For any $t \notin T_0 = \bigcup_{n=1}^{\infty} T_n$ we have $m(M_t) = 0$; consequently, $M_t \perp M_0$ and $\bigoplus_{t \notin T_0} M_t \perp M_0$. Hence, $m(\bigoplus_{t \notin T_0} M_t) = 0$ and

$$m(M) = m(\textstyle\bigoplus_{t \in T_0} M_t) + m(\bigoplus_{\notin T_0} M_t) = \sum_{t \in T_0} m(M_t) + \sum_{t \notin T_0} m(M_t) \quad ■$$

Corollary 7. An inner product space V is complete iff there is a unit vector $e \in V$ such that the equality

$$\left\| \overline{\bigoplus_{n=1}^{\infty}} M_n e \right\|^2 = \sum_{n=1}^{\infty} \|\overline{M}_n e\|^2$$

holds for any sequence of orthogonal elements $\{M_n\}_{n=1}^{\infty} \subset \mathscr{L}(V)$.

Proof. The function $m_e\colon M \mapsto \|\overline{M}e\|^2$, $M \in \mathscr{L}(V)$, is a state on $\mathscr{L}(V)$ with a carrier P_e. ■

We note that, according to Maeda's theorem (Maeda, 1980), any totally additive state on $\mathscr{L}(V)$ of a complete inner product space V has a support.

3. GLEASON'S THEOREM AND σ-FINITE MEASURES

We show that on $\mathscr{L}(V)$ of any incomplete separable inner product space V there are many σ-finite measures, and we give their characterization via Gleason's formula for infinite measures.

We recall that a measure m on $\mathscr{L}(V)$ is σ-finite if there is a sequence of mutually orthogonal elements of $\mathscr{L}(V)$, $\{M_n\}_{n=1}^{\infty}$, such that $V=\bigoplus_{n=1}^{\infty} M_n$ and $m(M_n)<\infty$ for any $n\geq 1$.

The mapping $m: \mathscr{L}(V)\to[0,\infty]$ defined by $m(M)=\dim M$, $M\in\mathscr{L}(V)$, is a σ-finite measure on $\mathscr{L}(V)$. Moreover, if T is a continuous, positive, Hermitian operator on $\bar{V}$ with purely point spectrum consisting of countably many nonzero points of infinite multiplicity, then a mapping m_T defined via

$$m_T(M)=\operatorname{tr}(T\bar{M}), \qquad M\in\mathscr{L}(V)$$

is a σ-finite measure. Hence, the existence of σ-finite measures does not entail the completeness of V.

Theorem 8. Any σ-finite measure m on $\mathscr{L}(V)$ of a separable inner product space has a carrier.

Proof. Denote

$$M_f=\{x\in V: m(P_x)<\infty\}\cup\{0\} \tag{9}$$

$$M_0=\{x\in V: m(P_x)=0\}\cup\{0\} \tag{10}$$

Due to the Lugovaja-Sherstnev lemma (Lugovaja and Sherstnev, 1980), M_f and M_0 are linear submanifolds of V. Moreover, if $\{x_n\}\subset M_0$ and $x_n\to x\in V$, then $x\in M_0$. Indeed, using the Gram-Schmidt orthogonalization process to $\{x_n\}$, we may find mutually orthogonal vectors $\{e_n\}\subset M_0$ such that $\bigvee_{i=1}^{n} P_{x_i}=\bigoplus_{i=1}^{n} P_{e_i}$ for any n. Hence,

$$m\left(\bigvee_{i=1}^{\infty} P_{x_i}\right)=\lim_n m\left(\bigoplus_{i=1}^{n} P_{e_i}\right)=0$$

The closedness of $\bigvee_{n=1}^{\infty} P_{x_n}$ entails $x\in\bigvee_{n=1}^{\infty} P_{x_n}\subset M_0$.

Inasmuch as M_0 is also a separable inner product space, it contains an orthonormal basis $\{y_i\}$. We assert $M_0=\bigoplus_i P_{y_i}$. Indeed, let $N_0=\bigoplus_i P_{y_i}$ and let $y\in M_0$, $v\perp N_0$. Then

$$(y, v)=\sum_i (y, y_i)(y_i, v)=0$$

which gives $y\in N_0^{\perp\perp}=N_0$, so that $M_0\subset N_0$. Since for P_x the orthomodular property holds, we conclude $x\in M_0$, which entails $N_0\subset M_0$.

Hence, the element $M_0^{\perp}$ is a carrier of m. ■

Lemma 9. If m is a measure on $\mathscr{L}(V)$ of a separable inner product space V, then m is monotone on $\mathscr{L}(V)$.

Proof. Let $A \subset B$. If $m(B)=\infty$, then $m(A) \leq m(B)$ trivially. Suppose now $m(B)<\infty$. Due to the separability of V, A contains an orthonormal basis of A, $\{g_n\}$. Put $A_n = \bigoplus_{i=1}^n P_{g_i}$, then $A = \bigoplus_n P_{g_n}$ and

$$m(A) = \sum_n m(P_{g_n}) = \lim_n m(A_n)$$

For any finite-dimensional A_n we have

$$B = A_n(B \wedge A_n^{\perp}), \qquad n \geq 1$$

which implies $m(B) = m(A_n) + m(B \cap A_n^{\perp})$. Hence,

$$m(B) = m(A) + \lim_n m(B \cap A_n^{\perp}) \tag{11}$$

which yields $m(A) \leq m(B)$. ■

To formulate our result, we need the following notions. Let $B \in \mathscr{L}(V)$. By $\mathrm{Tr}(\bar{B})$ we denote the class of all bounded operators $T: \bar{B} \to \bar{B}$ such that, for every orthonormal basis $\{x_n\}$ of $\bar{B}$, the series $\sum_n (Tx_n, x_n)$ converges and is independent of the basis used; the expression $\mathrm{tr}\, T := \sum_n (Tx_n, x_n)$ is called the trace of T.

A bilinear form is a function $t: D(t) \times D(t) \to C$ [$D(t)$ is a linear submanifold in V called the domain of the definition of t] such that t is linear in both arguments, and $t(\alpha x, \beta y) = \alpha\bar{\beta} t(x, y)$, $x, y \in D(t)$, $\alpha, \beta \in C$. If $t(x, y) = \overline{t(y, x)}$ for all $x, y \in D(t)$, then t is said to be symmetric; if for a symmetric bilinear form t we have $t(x, x) \geq 0$ for all $x \in D(t)$, then t is said to be positive. Let $B \in \mathscr{L}(V)$ and let $B \subset D(t)$. Then, by $t \between B$ we mean a symmetric bilinear form with $D(t \between B) = B$ defined by $t \between B(x, y) = t(x, y)$, $x, y \in B$. If $t \between B$ is induced by a trace operator $T \in \mathrm{Tr}\, \bar{B}$, that is, $t \between B(x, y) = (Tx, y)$ for all $x, y \in B$, then we say $t \between B \in \mathrm{Tr}(\bar{B})$ and we put $\mathrm{tr}(t \between B) = \mathrm{tr}\, T$.

Lemma 10. Let m be a measure of a separable inner product space V. Let $m(B) < \infty$ and $\dim B \geq 3$. Then there is a unique positive Hermitian operator $T_B: \bar{B} \to \bar{B}$ such that $T_B \in \mathrm{Tr}(\bar{B})$ and

$$m(A) = \mathrm{tr}(T_B \bar{A}), \qquad A \in \mathscr{L}(V), \qquad A \subset B \tag{12}$$

Proof. Since M_f defined by (10) is a linear submanifold of V, then, using the Gleason theorem for finite-dimensional subspaces of finite measure, we conclude that the function $\varphi: B \to [0, \infty]$ defined via

$$\varphi(f) = \begin{cases} m(P_f) & \text{if } \ 0 \neq f \in B \\ 0 & \text{if } \ f = 0 \end{cases}$$

is induced by some bounded, positive, symmetric, bilinear form t_B, with $D(t_B) = B$. Therefore, there is a unique, positive, Hermitian operator $T_B: \bar{B} \to \bar{B}$ such that

$$\varphi(f) = t_B(f, f) = (T_B f, f), \qquad f \in B$$

Inasmuch as V is separable, B contains at least one orthonormal basis of B, $\{f_n\}$, which is also the orthonormal basis of $\bar{B}$. Hence,

$$
\begin{aligned}
m(B) &= m\left(\bigoplus_n P_{f_n}\right) = \sum_n m(P_{f_n}) \\
&= \sum_n \varphi(f_n) = \sum_n (T_B f_n, f_n) \\
&= \mathrm{tr}(T_B \bar{B}) < \infty
\end{aligned}
$$

which entails $T_B \in \mathrm{Tr}(\bar{B})$.

Let now $A \subset B$, $A \in \mathscr{L}(V)$. According to Lemma 9, $m(A) < \infty$. Let us choose an orthonormal basis of A, $\{g_n\}$; then

$$m(A) = \sum_n m(P_{g_n}) = \sum_n (T_B g_n, g_n) = \mathrm{tr}(T_B \bar{A}) \quad \blacksquare$$

The following Gleason theorem for σ-finite measures on $\mathscr{L}(V)$ of a separable inner product space V has been proved by Lugovaja and Sherstnev (1980) and generalized to nonseparable Hilbert spaces by Dvurečenskij (1985, 1986a).

Theorem 11. (Gleason's theorem). Let m be a σ-finite measure on $\mathscr{L}(V)$ of a separable inner product space V. Then there is a unique symmetric, positive, bilinear form t with $D(t)^{\perp\perp} = V$ such that

$$m(B) = \begin{cases} \mathrm{tr}(t \Phi B) & \text{iff} \quad t \Phi B \in \mathrm{Tr}(\bar{B}) \\ \infty & \text{otherwise} \end{cases} \tag{13}$$

Proof. Using Lemma 10, the system of positive, symmetric, bilinear forms $\{t_B \colon B \in \mathscr{L}(V),\, m(B) < \infty\}$ defines a unique symmetric bilinear form t with $D(t) = M_f$ via

$$t(f, f) = t_B(f, f) = m(P_f), \qquad f \in M_f$$

Suppose $m(B) < \infty$. In view of Lemma 10, there exists a positive Hermitian operator $T_B \colon \bar{B} \to \bar{B}$ such that (12) holds.

Conversely, let $t \Phi B \in \mathrm{Tr}(\bar{B})$. Hence, $B \subset M_f$ and, for an orthonormal basis $\{f_n\}$ of B, we have

$$
\begin{aligned}
m(B) &= \sum_n m(P_{f_n}) = \sum_n t(f_n, f_n) \\
&= \sum_n t \Phi B(f_n, f_n) = \mathrm{tr}(t \Phi B) \\
&< \infty \quad \blacksquare
\end{aligned}
$$

We do not know whether the Gleason and Maeda theorems hold for nonseparable inner product spaces.

Corollary 12. Let m be a σ-finite measure on $\mathscr{L}(V)$ of a separable inner product space V. Let $0 < m(B) < \infty$ for some $B \in \mathscr{L}(V)$. Then there is a unit vector $e \in \bar{B}$ such that the mapping m_e: $\mathscr{L}_B := \{A \in \mathscr{L}(V): A \subset B\} \to [0, 1]$ defined via

$$m_e(A) = \|\bar{A}e\|^2, \qquad A \in \mathscr{L}_B \tag{14}$$

is a σ-additive function on $\mathscr{L}_B$.

Proof. Due to Lemma 10, there is a unique positive Hermitian operator $T_B \in \mathrm{Tr}(\bar{B})$ such that (12) holds. Therefore, T_B is of the form $T_B = \sum_n \lambda_n(\cdot, f_n)f_n$, where $\lambda_n > 0$ and $f_n \in \bar{B}$. The mappings m_n: $A \mapsto \|\bar{A}f_n\|^2$, $A \in \mathscr{L}_B$, satisfy the inequality (3). We assert that in (3) the equality holds for any n. Indeed, if not, then for some $\bigoplus_i M_i$ we have

$$m\left(\bigoplus_i M_i\right) = \sum_n \lambda_n m_n \left(\bigoplus_i M_i\right) > \sum_m \lambda_n \sum_i m_n(M_i)$$

$$= \sum_i \sum_n \lambda_n m_n(M_i) = \sum_i m(M_i) = m\left(\bigoplus_i M_i\right) \quad \blacksquare$$

The problem: If $0 < m(B) < \infty$, is then B complete?

REFERENCES

Amemiya, I., and Araki, M. (1966). A remark on Piron's paper, *Publications of the Research Institute for Mathematical Sciences, Series A*, **2**, 423-427.

Drisch, T. (1979). Generalization of Gleason's theorem, *International Journal of Theoretical Physics*, **18**, 239-243.

Dvurečenskij, A. (1985). Gleason theorem for signed measures with infinite values, *Mathematica Slovaca*, **35**, 319-325.

Dvurečenskij, A. (1986a). Generalization of Maeda's theorem, *International Journal of Theoretical Physics*, **25**, 1117-1124.

Dvurečenskij, A. (1986b). Note on a construction of unbounded measures on a nonseparable Hilbert space quantum logic, Submitted for publication.

Dvurečenskij, A. (1987). Converse of the Eilers-Horst theorem, *International Journal of Theoretical Physics*, **26**, 609-612.

Eilers, M., and Horst, E. (1975). The theorem of Gleason for nonseparable Hilbert space, *International Journal of Theoretical Physics*, **13**, 419-424.

Gleason, A. M. (1957). Measures on the closed subspaces of a Hilbert space, *Journal of Mathematics and Mechanics*, **6**, 369-375.

Gudder, S. P. (1974). Inner product spaces, *American Mathematical Monthly*, **81**, 29-36.

Gudder, S. P. (1975). Correction to: "Inner product spaces," *American Mathematical Monthly*, **82**, 251.

Gudder, S. P., and Holland, S. (1975). Second correction to: Inner product spaces, *American Mathematical Monthly*, **82**, 818.

Hamhalter, J., and Pták, P. (1987). A completeness criterion for inner product spaces, *Bulletin of the London Mathematical Society*, **19**, 259-263.

Holland, S. (1969). Partial solution to Mackey's problem about modular pairs and completeness, *Canadian Journal of Mathematics*, **21**, 1518–1525.
Lugovaja, G. D., and Sherstnev, A. N. (1980). On the Gleason theorem for unbounded measures, *Izvestija Vuzov Matematika*, **1980**(12), 30–32 (in Russian).
Maeda, S. (1980). *Lattice Theory and Quantum Logic*, Mahishoten, Tokyo (in Japanese).
Ulam, S. (1930). Zur Masstheorerie in der allgemeinen Mengelehre, *Fundamenta Mathematicae*, **16**, 140–150.